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LETTER TO THE EDITOR

About the critical condition of the q-state Potts model on the anisotropic hypercubic lattice

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Abstract. A critical condition is obtained for the ferromagnetic model on the hypercubic lattice in d dimensions with different Potts interactions along the different lattice axes. It is done by extending a simple procedure which is shown to be exact for the square lattice in $d=2$ dimensions.

The model considered in this letter is the q-state Potts ferromagnet with different two-site interactions along the different lattice axes. For such a Potts model on a lattice of N sites, the Hamiltonian $\mathscr{H}$ generally takes the form

$$\mathscr{H} = -\sum_{\alpha=1}^{p} \varepsilon_\alpha \sum_{\langle ij\rangle} \delta_{kr}(\sigma_i, \sigma_j). \tag{1}$$

Here $\sigma_i = 1, 2, \ldots, q$ specifies the spin state at the ith site, $\delta_{kr}(.,.)$ is the Kronecker delta, $\varepsilon_\alpha > 0$ is the strength of the two-site interaction along the α axis ($\alpha = 1, 2, 3, \ldots, p$) and the sum is taken over the nearest-neighbour sites on the lattice. The exact critical condition for this model is known only in $d=2$ dimensions for the square, triangular and honeycomb lattice (Potts 1952, Kihara *et al* 1954, Suzuki and Fisher 1971, Kim and Joseph 1974, Baxter *et al* 1978, Burkhardt and Southern 1978, Hinterman *et al* 1978). (A review of all these results may be found in Wu (1982).) To our best knowledge there are no results available for anisotropic models in higher than two dimensions.

In the present letter we are going to make two steps. In the first step we present a simple procedure which produces the known exact results for the $q=2$ model in $d=2$ dimensions. After that we extend to the anisotropic case a critical condition which was previously obtained (Hajduković 1983) for the isotropic model on the hypercubic lattice in d dimensions.

To this end let us rewrite the Hamiltonian (1) in the form

$$\mathscr{H} = -\sum_{\alpha=1}^{p} \varepsilon_\alpha n_\alpha \tag{2}$$

where n_α is the number of bonds along the α axis ($\alpha = 1, 2, \ldots, p$) with both ends in the same state. The partition function is then

$$Z = \sum_{\{n\}} G(n_1, n_2, \ldots, n_p) \exp\left(\sum K_\alpha n_\alpha\right) \equiv \sum_{\{n\}} z(n_1, n_2, \ldots, n_p). \tag{3}$$

Here $K_\alpha \equiv \varepsilon_\alpha / k_B T$, the sum is taken over all possible values of $n_1, n_2, \ldots, n_p$ and $G(n_1, n_2, \ldots, n_p)$ is the number of configurations for a given sequence $n_1, n_2, \ldots, n_p$.

To be definite let us consider the case of a square lattice with couplings K_1 and K_2 along two different lattice axes. It is extremely difficult to obtain $z(n_1, n_2)$ for the entire lattice. They are, however, easily obtained for just a simple square. The assumption is that the information about the critical condition of the *infinite* system is retained in the values of $z(n_1, n_2)$ for a *single* square. The possible values of $z(n_1, n_2)$ for a single square in the case $q = 2$ are

$$\begin{aligned} z(2,2) &= 2\exp[2(K_1+K_2)] \\ z(1,1) &= 8\exp(K_1+K_2) \\ z(2,0) &= 2\exp(2K_1) \\ z(0,2) &= 2\exp(2K_2) \\ z(0,0) &= 2. \end{aligned} \tag{4}$$

The values (4) can be classified into three sets: S_4, S_2 and S_0 which are determined by the conditions

$$n \equiv n_1 + n_2 = 4, 2, 0$$

respectively. In the isotropic case ($K_1 = K_2$) every set determined in such a way corresponds to a definite 'energy level' of the square considered. For $n = 0, 4$ the spin state of a spin uniquely determines the states of the other spins on the square. For $n = 2$ the knowledge of the state of a single spin is not sufficient to determine the states of the others. In a way, sets S_4 and S_0 represents the states with rigorous regularity in order, while S_2 represent the states with no regularity in order. Let us assume that, at the critical temperature of an infinite lattice, the sum of values from sets S_4 and S_0 is equal to the sum of values from S_2, i.e. that the critical condition is

$$z(2,2) - z(1,1) - z(2,0) - z(0,2) + z(0,0) = 0. \tag{5}$$

If we introduce equation (4), equation (5) becomes

$$2[\exp(K_1+K_2) - \exp(K_1) - \exp(K_2) - 1][\exp(K_1+K_2) + \exp(K_1) + \exp(K_2) - 1] = 0 \tag{6}$$

i.e. because we are interested only in the solution with $K_1, K_2 > 0$,

$$\exp(K_1+K_2) - \exp(K_1) - \exp(K_2) - 1 = 0. \tag{7}$$

This is the well known exact critical condition for the $q = 2$ ferromagnetic Potts model on the square lattice.

As another example let us consider the triangular lattice with couplings K_1, K_2, K_3 along three different lattice axes. The possible values of $z(n_1, n_2, n_3)$ for a single triangle in the case $q = 2$ are

$$\begin{aligned} z(1,1,1) &= 2\exp(K_1+K_2+K_3) \\ z(1,0,0) &= 2\exp(K_1) \\ z(0,1,0) &= 2\exp(K_2) \\ z(0,0,1) &= 2\exp(K_3). \end{aligned} \tag{8}$$

The same procedure as in the case of the square lattice gives the exact critical condition

$$z(1,1,1) - z(1,0,0) - z(0,1,0) - z(0,0,1) = 0$$

i.e.

$$\exp(K_1+K_2+K_3)-\exp(K_1)-\exp(K_2)-\exp(K_3)=0. \tag{9}$$

For a honeycomb lattice it is possible to obtain the exact critical condition by considering a single hexagon or, which is simpler, a site of the honeycomb lattice with its three neighbours. In the latter case, for $q=2$ we have the possible values

$$\begin{aligned}
z(1,1,1)&=2\exp(K_1+K_2+K_3)\\
z(1,1,0)&=2\exp(K_1+K_2)\\
z(1,0,1)&=2\exp(K_1+K_3)\\
z(0,1,1)&=2\exp(K_2+K_3)\\
z(1,0,0)&=2\exp(K_1)\\
z(0,1,0)&=2\exp(K_2)\\
z(0,0,1)&=2\exp(K_3)\\
z(0,0,0)&=2
\end{aligned} \tag{10}$$

and the same rule as for the square and triangular lattice leads to

$$\begin{aligned}
&z(1,1,1)-z(1,1,0)-z(1,0,1)-z(0,1,1)\\
&\qquad -z(1,0,0)-z(0,1,0)-z(0,0,1)+z(0,0,0)=0
\end{aligned}$$

i.e.

$$\begin{aligned}
&\exp(K_1+K_2+K_3)-\exp(K_1+K_2)-\exp(K_1+K_3)\\
&\qquad -\exp(K_2+K_3)-\exp(K_1)-\exp(K_2)-\exp(K_3)+1=0
\end{aligned} \tag{11}$$

which is again the exact critical condition.

We do not know why the procedure described gives the exact results but it is obviously not a coincidence. In our opinion further understanding of this procedure will be important.

We now turn to the problem of the determination of the critical condition for the anisotropic model on the hypercubic lattice in d dimensions. Let us point out two facts about the critical condition of a square lattice (i.e. a hypercubic lattice in $d=2$ dimensions).

(i) The exact critical condition for general q differs from (7) only in the right-hand side which is equal to $q-2$ instead of zero. So the exact critical condition is of the form

$$F(K_1, K_2)=q-2.$$

The function $F(K_1, K_2)$ may be obtained by our procedure as a linear combination of the $z(n_1, n_2)$ for a single hypercube in $d=2$ dimensions (i.e. a single square).

(ii) The critical condition for the anisotropic model reduces to that of the isotropic model if we take $K_1=K_2\equiv K$, i.e. the critical condition for the isotropic model is to be obtained as a particular case ($K_1=K_2$) of the anisotropic model.

As a basis for our further considerations we accept that (i) and (ii) are true in the general case of d dimensions. However, in $d\geqslant 3$ dimensions it is not clear how to use rule (i). So, in the case of a simple cubic lattice with couplings K_1, K_2, K_3 along three lattice axes, by inspection of 256 different configurations of a *simple* cube for $q=2$, we may obtain the possible values of $z(n_1, n_2, n_3)$ but we do not know an actual rule to form the linear combination needed with these values of $z(n_1, n_2, n_3)$. At this point rule (ii) can be helpful to us, because in a previous letter (Hajduković 1983) an approximate critical condition for the q-state Potts model on the isotropic hypercubic lattice, which is in excellent agreement with numerical studies, was obtained. Thus, we are looking for such a critical condition of the anisotropic model which reduces to this known critical condition of the isotropic model. A simple analysis of all possible values of $z(n_1, n_2, n_3)$ for a single cube shows that for $q = 2$ this requirement is satisfied only if

$$z(3,3,3) - z(2,2,2) + z(1,1,1) = 0. \tag{12}$$

For a single cube we have

$$\begin{aligned} z(3,3,3) &= 16\exp[3(K_1 + K_2 + K_3)] \\ z(2,2,2) &= 64\exp[2(K_1 + K_2 + K_3)] \\ z(1,1,1) &= 16\exp(K_1 + K_2 + K_3) \end{aligned} \tag{13}$$

so that (12) may be transformed into

$$\exp(K_1 + K_2 + K_3)\{\exp(K_1 + K_2 + K_3) - \sqrt{2}\exp[(K_1 + K_2 + K_3)/2] - 1\} \\ \times \{\exp(K_1 + K_2 + K_3) + \sqrt{2}\exp[(K_1 + K_2 + K_3)/2] - 1\} = 0$$

i.e. because we are interested only in the solution with $K_1, K_2, K_3 > 0$,

$$\exp(K_1 + K_2 + K_3) - \sqrt{2}\exp[(K_1 + K_2 + K_3)/2] - 1 = 0. \tag{14}$$

Thus, in order to satisfy (i), (ii) and the condition for the isotropic model (Hajduković 1983) the critical condition for the anisotropic model on the SC lattice is to be

$$\exp\left(\sum_{\alpha=1}^{3} K_\alpha\right) - \sqrt{2}\exp\left[\frac{1}{2}\left(\sum_{\alpha=1}^{3} K_\alpha\right)\right] - 1 = q - 2 \tag{15}$$

and for the general case of d dimensions

$$\exp\left(\sum_{\alpha=1}^{d} K_\alpha\right) - 2^{1/(d-1)}\exp\left[\frac{1}{2}\left(\sum_{\alpha=1}^{d} K_\alpha\right)\right] - 1 = q - 2. \tag{16}$$

So, we think there are good reasons to conclude that (15), and especially (14), are reliable critical conditions for the corresponding lattices. From (15) we have the prediction

$$K_{1c} + K_{2c} + K_{3c} + \ldots + K_{dc} \equiv dK_c \tag{17}$$

i.e. for the SC lattice

$$K_{1c} + K_{2c} + K_{3c} = \ln[q + (2q-1)^{1/2}] \equiv 3K_c \tag{18}$$

where K_c is the critical coupling for the corresponding isotropic lattice and $K_{\alpha c}$ $(\alpha = 1, 2, \ldots)$ are critical couplings of the anisotropic model.

At this point a comparison with numerical results in the anisotropic case would be welcome. Unfortunately, we have not been able to find such results in the literature about Potts models.

In conclusion, let us point out a difference between the isotropic and anisotropic cases. In the isotropic case, the critical condition (Hajduković 1983) is an equation of the same structure for *all d*. In fact, it is equation (16) with $K_1 = K_2 = \ldots = K_d \equiv K$. However, in the anisotropic case, with $d = 2$, (16) does not agree with (7). Thus, in a way, our procedure leads to the conclusion that in the anisotropic case the critical condition for $d=2$ and $d \geqslant 3$ are equations with different structure. It is interesting to note that it is in agreement with the rigorous result of Maillard and Rammal (1983) that the unknown critical condition in $d=3$ dimensions cannot be an equation of the same structure as in $d=2$ dimensions.

Besides the lattices considered in this letter it may be of interest to apply the same technique to other lattices such as the generalised square lattice, kagome lattice and the 3-12 lattice. We hope to be able to present corresponding results for some other lattices in the near future.

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